An extension of the extended partial fraction theorem

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The extended partial fraction theorem can be found in most algebra books as follows :

"If g(x) is a polynomial of degree < n, over the same field $a_1, a_2, \ldots., a_n$, then, over this field,

$$\frac{g(x)}{f(x)} = \sum_{r=1}^{n} \frac{g(a_r)}{f'(a_r)} \frac{1}{x - a_r}$$

provided that $a_1, a_2, \ldots., a_n$ are unequal."

The condition that $a_1, a_2, \ldots., a_n$ must be distinct limits the application of the above theorem. The aim of this article is to extend it to the case that repeated linear factors of f(x) are allowed.

First, we let :

$$\frac{g(x)}{(x-a)p(x)} \equiv \frac{A}{x-a} + \frac{B(x)}{p(x)} \qquad (1)$$

where (x – a) is not a factor of p(x).

Then $\quad g(x) = A\, p(x) + (x - a)\, B(x) \qquad (2)$

Putting x = a in (2), then $g(a) = A\, p(a)$

and $\quad A = \frac{g(a)}{p(a)} \qquad (3)$

On differentiating both sides of (1) partially (n – 1)th times with respect to a,

$$\frac{\partial^{n-1}}{\partial a^{n-1}}\left[\frac{g(x)}{(x-a)p(x)}\right] = \frac{\partial^{n-1}}{\partial a^{n-1}}\left[\frac{A}{x-a}\right] + \frac{\partial^{n-1}}{\partial a^{n-1}}\left[\frac{B(x)}{p(x)}\right] \qquad (4)$$

Leaving aside any evaluation of the last term in (4), and applying Leibnitz' theorem on the first term on the right hand side, we get :

$$\frac{g(x)}{p(x)}\frac{\partial^{n-1}}{\partial a^{n-1}}\left[\frac{1}{x-a}\right]=\sum_{r=0}^{n-1}C_{n-1}^{r}\frac{\partial^{r}A}{\partial a^{r}}\frac{\partial^{n-r-1}}{\partial a^{n-1-r}}\left[\frac{1}{x-a}\right]+\cdots \tag{5}$$

Putting $A^{(r)}$ to be the r th partial derivative of A with respect to a, and $A^{(0)} = A$, then (5) becomes :

$$\frac{(n-1)!}{(x-a)^{n}}\frac{g(x)}{p(x)}=\sum_{r=0}^{n-1}C_{n-1}^{r}A^{(r)}\left[\frac{1}{x-a}\right]^{(n-1-r)}+\cdots$$

$$=\sum_{r=0}^{n-1}\frac{(n-1)!}{r!(n-1-r)!}A^{(r)}\frac{(n-1-r)!}{(x-a)^{n-r}}+\cdots$$

$$\therefore\quad \frac{g(x)}{(x-a)^{n}p(x)}=\sum_{r=0}^{n-1}\frac{A^{(r)}}{r!}\frac{1}{(x-a)^{n-r}}+\cdots \tag{6}$$

From (6) we can see that the corresponding coefficients for the partial fractions

$\frac{1}{(x-a)^{n}},\ \frac{1}{(x-a)^{n-1}},\cdots,\frac{1}{x-a}$ are respectively :

$$A,\ \frac{1}{1!}\frac{\partial A}{\partial a},\frac{1}{2!}\frac{\partial^{2}A}{\partial a^{2}},\ \ldots.,\ \frac{1}{(n-1)!}\frac{\partial^{n-1}A}{\partial a^{n-1}} \tag{7}$$

So far, we have not got any partial fractions for other factors of p(x). However, the evaluation is essentially similar for these factors.

$$\therefore\quad h(x)\equiv\frac{g(x)}{(x-a_1)^{n_1}(x-a_1)^{n_2}\cdots(x-a_1)^{n_k}}$$

$$=\sum_{s=0}^{k}\sum_{r=0}^{n_s-1}\frac{{A_s}^{(r)}}{r!}\frac{1}{(x-a_s)^{n_s-r}} \tag{8}$$

where A_s is the substitution of $x = a_s$ in $h(x)(x-a_s)^{n_s}$.

Example

Express $E \equiv \dfrac{x^4}{(x-1)^3(x-2)^2}$

Putting $a_1 = a = 1$

$$A_1 = \frac{a^4}{(a-2)^2} = 1$$

$$\frac{1}{1!}A_1^{(1)} = \frac{2a^4 - 8a^3}{(a-2)^3} = 6$$

$$\frac{1}{2!}A_1^{(2)} = \frac{1}{2!}\frac{2a^4 - 16a^3 + 48a^2}{(a-2)^4} = 17$$

And putting $a_2 = b = 2$,

$$A_2 = \frac{b^4}{(b-1)^3} = 16$$

$$\frac{1}{1!}A_2^{(1)} = \frac{b^4 - 4b^3}{(b-1)^4} = -16$$

From equation (8),

$$E = \frac{1}{(x-1)^3} + \frac{6}{(x-1)^2} + \frac{17}{x-1} + \frac{16}{(x-2)^2} - \frac{16}{x-2}$$

References :

(1) Algebra, J.W. Archbold.
(2) Higher Algebra for School, W.L. Ferrar.
(3) Techniques of Mathematical Analysis, C. J. Tranter.